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Gaudin hypothesis for the XYZ spin chain

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Abstract. The XYZ spin chain is considered within the framework of the generalized algebraic Bethe ansatz developed by Takhtajan and Faddeev. The sum of norms of the Bethe vectors is computed and expressed in the form of a Jacobian. This result corresponds to the Gaudin hypothesis for the XYZ spin chain.

1. Introduction

In this paper we consider the XYZ spin chain with the periodic boundary condition. The Hamiltonian is defined by

$$H_{XYZ} = -\tfrac{1}{2}\sum_{n=1}^{L}(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z). \tag{1.1}$$

Here σ_n^x, σ_n^y and σ_n^z are the Pauli matrices acting on a Hilbert state space $H_n = \mathbb{C}^2$. The Hamiltonian thus acts on $\otimes_{n=1}^L H_n$. The coupling constants J_x, J_y and J_z are parametrized by

$$J_x = 1 + k\,\mathrm{sn}^2\,2\eta \qquad J_y = 1 - k\,\mathrm{sn}^2\,2\eta \qquad J_z = \mathrm{cn}\,2\eta\,\mathrm{dn}\,2\eta \tag{1.2}$$

where k is the modulus of the Jacobi elliptic functions. In the limit $k \to 0$ J_x, J_y and J_z satisfy $J_x = J_y = 1$ and $J_z = \cos 2\eta$, and the XYZ spin chain is reduced to the XXZ spin chain.

The XYZ spin chain was first solved by Baxter in a series of remarkable papers [1, 2]. He discovered a link between the XYZ spin chain and a two-dimensional classical model, the so-called eight-vertex model, and obtained a system of transcendental equations. With the help of these equations the energy of the ground state of the XYZ spin chain was calculated. Furthermore, he found the eigenvectors and eigenvalues of the XYZ spin chain by means of a generalization of the Bethe ansatz method [3]. Referring to the algebraic Bethe ansatz, which is more intelligible than the Bethe ansatz, Takhtajan and Faddeev succeeded in simplifying Baxter's method [4]. Their method is called the *generalized algebraic Bethe ansatz* and enables us to deal with the XYZ spin chain more systematically.

For the XXZ spin chain, by means of the usual algebraic Bethe ansatz, various correlation functions have been calculated. Gaudin forecasted that norms of the eigenvectors are expressed by Jacobians, and Korepin proved his hypothesis [5]. Based on this fact scalar products of arbitrary vectors were shown to be represented by determinants of matrices that contain bosonic quantum fields called the dual fields [6]. Using them one can evaluate any correlation function

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of the XXZ spin chain. Recently, these results have been extended to the asymmetric XXZ chain that is a non-Hermitian generalization of the XXZ spin chain [7].

The aim of the paper is to prove the Gaudin hypothesis for the XYZ spin chain by using the generalized algebraic Bethe ansatz. We show that the sum of norms of the Bethe vectors is expressed by a Jacobian. However, norms of the eigenvectors cannot be computed within the framework of the generalized algebraic Bethe ansatz, because the Bethe vectors are not equivalent to the eigenvectors (see (2.42), (2.43) and (2.55), (2.56)). We interpret the Gaudin hypothesis as a theorem that holds for the Bethe vectors. This interpretation is supported by the fact that the Bethe vectors correspond to the eigenvectors in the usual algebraic Bethe ansatz for the XXZ spin chain.

Our result lays the foundation for the calculation of correlation functions of the XYZ spin chain. In section 2 we review the generalized algebraic Bethe ansatz. In section 3 the sum of norms of the Bethe vectors is shown to be given by a Jacobian. Section 4 is devoted to concluding remarks.

2. Generalized algebraic Bethe ansatz

In this section we review the generalized algebraic Bethe ansatz for the XYZ spin chain. In the original paper [4] the dual eigenvectors were not investigated. We include them for the first time.

2.1. Description of the model

Central objects of the generalized algebraic Bethe ansatz are the R-matrix and the L-operator. The R-matrix is of the form

$$R(\lambda,\mu) = \begin{pmatrix} a(\lambda,\mu) & 0 & 0 & d(\lambda,\mu) \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ d(\lambda,\mu) & 0 & 0 & a(\lambda,\mu) \end{pmatrix} \tag{2.1}$$

where

$$\begin{aligned} a(\lambda,\mu) &= \Theta(2\eta)\Theta(\lambda-\mu)H(\lambda-\mu+2\eta) \\ b(\lambda,\mu) &= H(2\eta)\Theta(\lambda-\mu)\Theta(\lambda-\mu+2\eta) \\ c(\lambda,\mu) &= \Theta(2\eta)H(\lambda-\mu)\Theta(\lambda-\mu+2\eta) \\ d(\lambda,\mu) &= H(2\eta)H(\lambda-\mu)H(\lambda-\mu+2\eta). \end{aligned} \tag{2.2}$$

We call $\lambda, \mu \in \mathbb{C}$ the spectral parameters. $H(\mu)$ and $\Theta(\mu)$ are the Jacobi theta functions with quasi-periods $2K, 2\mathrm{i}K' \in \mathbb{C}$ ($\mathrm{Im}\, \mathrm{i}K'/K > 0$). In this paper we assume that there exist $Q \in \mathbb{Z}_{>0}$ such that

$$Q\eta = 2K. \tag{2.3}$$

Then $H(\mu)$ and $\Theta(\mu)$ have a period $2Q\eta$:

$$H(\mu+2Q\eta) = H(\mu) \qquad \Theta(\mu+2Q\eta) = \Theta(\mu). \tag{2.4}$$

The L-operator is expressed by a 2×2 matrix whose elements contain the Pauli matrices:

$$L_n(\mu) = \begin{pmatrix} w_4 + w_3\sigma_n^z & w_1\sigma_n^x - \mathrm{i}w_2\sigma_n^y \\ w_1\sigma_n^x + \mathrm{i}w_2\sigma_n^y & w_4 - w_3\sigma_n^z \end{pmatrix} \tag{2.5}$$

where

$$\begin{aligned} w_4 + w_3 &= \Theta(2\eta)\Theta(\mu - \eta)H(\mu + \eta) \\ w_4 - w_3 &= \Theta(2\eta)H(\mu - \eta)\Theta(\mu + \eta) \\ w_1 + w_2 &= H(2\eta)\Theta(\mu - \eta)\Theta(\mu + \eta) \\ w_1 - w_2 &= H(2\eta)H(\mu - \eta)H(\mu + \eta). \end{aligned} \tag{2.6}$$

The L-operator $L_n(\mu)$ acts on a Hilbert state space H_n. This satisfies the *Yang–Baxter equation*:

$$R(\lambda, \mu)(L_n(\lambda) \otimes L_n(\mu)) = (L_n(\mu) \otimes L_n(\lambda))R(\lambda, \mu). \tag{2.7}$$

The Hamiltonian is derived from the L-operator as follows. The product of the L-operators is called the monodromy matrix and is expressed in a 2×2 matrix form:

$$T(\mu) = \overleftarrow{\prod_{n=1}^{L}} L_n(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ C(\mu) & D(\mu) \end{pmatrix}. \tag{2.8}$$

The trace of the monodromy matrix over matrix space

$$t(\mu) = \mathrm{tr}\, T(\mu) = A(\mu) + D(\mu) \tag{2.9}$$

is called the transfer matrix and gives the Hamiltonian (1.1) via

$$H_{XYZ} = -\,\mathrm{sn}\, 2\eta \frac{\mathrm{d}}{\mathrm{d}\mu} \log t(\mu)\bigg|_{\mu=\eta} + \text{constant}. \tag{2.10}$$

The Hamiltonian is thus diagonalized by the eigenvectors of the transfer matrix.

2.2. Gauge transformations

We introduce a family of gauge transformations with free parameters $s, t \in \mathbb{C}$ and integer $l = 0, \ldots, Q-1$. The L-operator is replaced by

$$L_n^l(\mu) = M_{n+l}^{-1}(\mu)L_n(\mu)M_{n+l-1}(\mu) = \begin{pmatrix} \alpha_n^l(\mu) & \beta_n^l(\mu) \\ \gamma_n^l(\mu) & \delta_n^l(\mu) \end{pmatrix} \tag{2.11}$$

with matrices $M_k(\mu)$ ($k = 0, \ldots, Q-1$) defined by

$$M_k(\mu) = \begin{pmatrix} H(s + 2k\eta - \mu) & (g(\tau_k))^{-1}H(t + 2k\eta + \mu) \\ \Theta(s + 2k\eta - \mu) & (g(\tau_k))^{-1}\Theta(t + 2k\eta + \mu) \end{pmatrix} \tag{2.12}$$

where

$$\tau_k = \tfrac{1}{2}(s + t) + 2k\eta - K \qquad g(\mu) = H(\mu)\Theta(\mu). \tag{2.13}$$

In the generalized algebraic Bethe ansatz the following vectors are important:

$$|\omega_n^l\rangle = H(s + (2(n+l) - 1)\eta)\,|\uparrow\rangle_n + \Theta(s + (2(n+l) - 1)\eta)\,|\downarrow\rangle_n \tag{2.14}$$

$$\langle\omega_n^l| = {}_n\langle\uparrow|\,\Theta(t + (2(n+l) - 1)\eta) - {}_n\langle\uparrow|\,H(t + (2(n+l) - 1)\eta). \tag{2.15}$$

Here $|\uparrow\rangle_n$ and $|\downarrow\rangle_n$ are the orthonormal basis of a Hilbert state space H_n, and ${}_n\langle\uparrow|$ and ${}_n\langle\downarrow|$ are those dual basis. The actions of elements of the transformed L-operator on $|\omega_n^l\rangle$ and $\langle\omega_n^l|$ are computed as follows:

$$\alpha_n^l(\mu)|\omega_n^l\rangle = h(\mu+\eta)|\omega_n^{l-1}\rangle \tag{2.16}$$

$$\delta_n^l(\mu)|\omega_n^l\rangle = h(\mu-\eta)|\omega_n^{l+1}\rangle \tag{2.17}$$

$$\gamma_n^l(\mu)|\omega_n^l\rangle = 0 \tag{2.18}$$

$$\langle\omega_n^l|\alpha_n^l(\mu) = \langle\omega_n^{l+1}|\frac{g(\tau_{n+l-1})}{g(\tau_{n+l})}h(\mu+\eta) \tag{2.19}$$

$$\langle\omega_n^l|\delta_n^l(\mu) = \langle\omega_n^{l-1}|\frac{g(\tau_{n+l})}{g(\tau_{n+l-1})}h(\mu-\eta) \tag{2.20}$$

$$\langle\omega_n^l|\beta_n^l(\mu) = 0 \tag{2.21}$$

where $h(\mu) = g(\mu)\Theta(0)$. Note that $|\omega_n^l\rangle$ and $\langle\omega_n^l|$ are independent of the spectral parameters. They are called the local vacuums.

For $k, l = 0, \ldots, Q-1$ we introduce a matrix

$$T_{k,l}(\mu) = M_k^{-1}(\mu)T(\mu)M_l(\mu) = \begin{pmatrix} A_{k,l}(\mu) & B_{k,l}(\mu) \\ C_{k,l}(\mu) & D_{k,l}(\mu) \end{pmatrix}. \tag{2.22}$$

Under the gauge transformations the monodromy matrix $T(\mu)$ is replaced by $T_{L+l,l}(\mu)$. We thus call $T_{k,l}(\mu)$ the generalized monodromy matrix. This plays a central role in the next subsection.

The products of the local vacuums are called the generating vectors:

$$|l\rangle = |\omega_L^l\rangle \otimes \cdots \otimes |\omega_1^l\rangle \qquad \langle l| = \langle\omega_1^l| \otimes \cdots \otimes \langle\omega_L^l|. \tag{2.23}$$

By use of local formulae (2.16)–(2.21) the actions of elements of the monodromy matrix on the generating vectors are computed as follows:

$$A_{L+l,l}(\mu)|l\rangle = (h(\mu+\eta))^L|l-1\rangle \tag{2.24}$$

$$D_{L+l,l}(\mu)|l\rangle = (h(\mu-\eta))^L|l+1\rangle \tag{2.25}$$

$$C_{L+l,l}(\mu)|l\rangle = 0 \tag{2.26}$$

$$\langle l|A_{L+l,l}(\mu) = \langle l+1|\frac{g(\tau_l)}{g(\tau_{L+l})}(h(\mu+\eta))^L \tag{2.27}$$

$$\langle l|D_{L+l,l}(\mu) = \langle l-1|\frac{g(\tau_{L+l})}{g(\tau_l)}(h(\mu-\eta))^L \tag{2.28}$$

$$\langle l|B_{L+l,l}(\mu) = 0. \tag{2.29}$$

If Q divides L extra factors of $g(\tau_l)$ and $g(\tau_{L+l})$ are cancelled. Hereafter we assume that the lattice length L is multiple of Q.

2.3. Generalized algebraic Bethe ansatz

The Yang–Baxter equation (2.7) can be shifted up to the level of the monodromy matrix:

$$R(\lambda,\mu)(T(\lambda)\otimes T(\mu)) = (T(\mu)\otimes T(\lambda))R(\lambda,\mu). \tag{2.30}$$

From this Yang–Baxter equation one can obtain the commutation relations among elements of the generalized monodromy matrix. Useful relations are the following:

$$A_{k,l}(\lambda)A_{k+1,l+1}(\mu) = A_{k,l}(\mu)A_{k+1,l+1}(\lambda) \tag{2.31}$$

$$B_{k,l+1}(\lambda)B_{k+1,l}(\mu) = B_{k,l+1}(\mu)B_{k+1,l}(\lambda) \tag{2.32}$$

$$C_{k+1,l}(\lambda)C_{k,l+1}(\mu) = C_{k+1,l}(\mu)C_{k,l+1}(\lambda) \tag{2.33}$$

$$D_{k+1,l+1}(\lambda)D_{k,l}(\mu) = D_{k+1,l+1}(\mu)D_{k,l}(\lambda) \tag{2.34}$$

$$A_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\lambda,\mu)B_{k,l-2}(\mu)A_{k+1,l-1}(\lambda) - \beta_{l-1}(\lambda,\mu)B_{k,l-2}(\lambda)A_{k+1,l-1}(\mu) \tag{2.35}$$

$$D_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\mu,\lambda)B_{k+2,l}(\mu)D_{k+1,l-1}(\lambda) + \beta_{k+1}(\lambda,\mu)B_{k+2,l}(\mu)D_{k+1,l-1}(\lambda) \tag{2.36}$$

$$C_{k-1,l-1}(\mu)A_{k,l}(\lambda) = \alpha(\lambda,\mu)A_{k+1,l-1}(\lambda)C_{k,l}(\mu) + \beta_k(\mu,\lambda)A_{k+1,l-1}(\mu)C_{k,l}(\lambda) \tag{2.37}$$

$$C_{k+1,l+1}(\mu)D_{k,l}(\lambda) = \alpha(\mu,\lambda)D_{k+1,l-1}(\lambda)C_{k,l}(\mu) - \beta_l(\mu,\lambda)D_{k+1,l-1}(\mu)C_{k,l}(\lambda) \tag{2.38}$$

$$\begin{aligned} C_{k-1,l+1}(\lambda)B_{k,l}(\mu) - \frac{g(\tau_{l-1})g(\tau_{l+1})}{g^2(\tau_l)}B_{k+1,l-1}(\mu)C_{k,l}(\lambda) \\ = \beta_k(\lambda,\mu)A_{k+1,l+1}(\lambda)D_{k,l}(\mu) - \beta_l(\lambda,\mu)A_{k+1,l+1}(\mu)D_{k,l}(\lambda) \end{aligned} \tag{2.39}$$

where

$$\alpha(\lambda,\mu) = \frac{h(\lambda-\mu-2\eta)}{h(\lambda-\mu)} \qquad \beta_k(\lambda,\mu) = \frac{h(2\eta)}{h(\mu-\lambda)}\frac{h(\tau_k+\mu-\lambda)}{h(\tau_k)}. \tag{2.40}$$

The generalized algebraic Bethe ansatz offers a simple method to find the eigenvectors and eigenvalues of the transfer matrix:

$$t(\mu) = \operatorname{tr} T(\mu) = A_{l,l}(\mu) + D_{l,l}(\mu). \tag{2.41}$$

Let us introduce vectors

$$|\Psi_l(\lambda_1,\ldots,\lambda_N)\rangle = B_{l+1,l-1}(\lambda_1)\cdots B_{l+N,l-N}(\lambda_N)|l-N\rangle \tag{2.42}$$

$$\langle\Psi_l(\lambda_1,\ldots,\lambda_N)| = \langle l-N+1|C_{l+N-1,l-N+1}(\lambda_N)\cdots C_{l,l}(\lambda_1). \tag{2.43}$$

Here we set

$$2N \equiv 0 \bmod Q. \tag{2.44}$$

Namely, the admissible values of N are

$$N = \begin{cases} 0, Q, 2Q, \ldots, L & \text{for odd } Q \\ 0, Q/2, Q, \ldots, L & \text{for even } Q. \end{cases}$$

Referring to the algebraic Bethe ansatz for the XXZ spin chain we call the vectors (2.42) and (2.43) the Bethe vectors. By means of commutation relations (2.32), (2.33), (2.35)–(2.38) and relations (2.24)–(2.29) the actions of $A_{l,l}(\mu)$ and $D_{l,l}(\mu)$ on the Bethe vectors are computed as follows:

$$\begin{aligned} A_{l,l}(\mu)|\Psi_l(\lambda_1,\ldots,\lambda_N)\rangle &= {}_1\Lambda(\mu;\{\lambda_k\})|\Psi_{l-1}(\lambda_1,\ldots,\lambda_N)\rangle \\ &+ \sum_{j=1}^{N} {}_1\Lambda_j^l(\mu;\{\lambda_k\})|\Psi_{l-1}(\lambda_1,\ldots,\lambda_{j-1},\mu,\lambda_{j+1},\ldots,\lambda_N)\rangle \end{aligned} \tag{2.45}$$

$$\begin{aligned} D_{l,l}(\mu)|\Psi_l(\lambda_1,\ldots,\lambda_N)\rangle &= {}_2\Lambda(\mu;\{\lambda_k\})|\Psi_{l+1}(\lambda_1,\ldots,\lambda_N)\rangle \\ &+ \sum_{j=1}^{N} {}_2\Lambda_j^l(\mu;\{\lambda_k\})|\Psi_{l+1}(\lambda_1,\ldots,\lambda_{j-1},\mu,\lambda_{j+1},\ldots,\lambda_N)\rangle \end{aligned} \tag{2.46}$$

$$\langle\Psi_{l-1}(\lambda_1,\ldots,\lambda_N)|A_{l,l}(\mu) = \langle\Psi_l(\lambda_1,\ldots,\lambda_N)|_1\Lambda(\mu;\{\lambda_k\}) + \sum_{j=1}^{N}\langle\Psi_l(\lambda_1,\ldots,\lambda_{j-1},\mu,\lambda_{j+1},\ldots,\lambda_N)|_1\widetilde{\Lambda}^l_j(\mu;\{\lambda_k\}) \qquad (2.47)$$

$$\langle\Psi_{l+1}(\lambda_1,\ldots,\lambda_N)|D_{l,l}(\mu) = \langle\Psi_l(\lambda_1,\ldots,\lambda_N)|_2\Lambda(\mu;\{\lambda_k\}) + \sum_{j=1}^{N}\langle\Psi_l(\lambda_1,\ldots,\lambda_{j-1},\mu,\lambda_{j+1},\ldots,\lambda_N)|_2\widetilde{\Lambda}^l_j(\mu;\{\lambda_k\}) \qquad (2.48)$$

where

$${}_1\Lambda(\mu;\{\lambda_k\}) = (h(\mu+\eta))^L\prod_{k=1}^{N}\alpha(\mu,\lambda_k) \qquad (2.49)$$

$${}_2\Lambda(\mu;\{\lambda_k\}) = (h(\mu-\eta))^L\prod_{k=1}^{N}\alpha(\lambda_k,\mu) \qquad (2.50)$$

$${}_1\Lambda^l_j(\mu;\{\lambda_k\}) = -\beta_{l-1}(\mu,\lambda_j)(h(\lambda_j+\eta))^L\prod_{k\neq j}^{N}\alpha(\lambda_j,\lambda_k) \qquad (2.51)$$

$${}_2\Lambda^l_j(\mu;\{\lambda_k\}) = \beta_{l+1}(\mu,\lambda_j)(h(\lambda_j-\eta))^L\prod_{k\neq j}^{N}\alpha(\lambda_k,\lambda_j) \qquad (2.52)$$

$${}_1\widetilde{\Lambda}^l_j(\mu;\{\lambda_k\}) = \beta_l(\lambda_j,\mu)(h(\lambda_j+\eta))^L\prod_{k\neq j}^{N}\alpha(\lambda_j,\lambda_k) \qquad (2.53)$$

$${}_2\widetilde{\Lambda}^l_j(\mu;\{\lambda_k\}) = -\beta_l(\lambda_j,\mu)(h(\lambda_j-\eta))^L\prod_{k\neq j}^{N}\alpha(\lambda_k,\lambda_j). \qquad (2.54)$$

For integer $m = 0,\ldots,Q-1$ consider the following linear combinations of the Bethe vectors:

$$|\Phi_m(\lambda_1,\ldots,\lambda_N)\rangle = \frac{1}{\sqrt{Q}}\sum_{l=0}^{Q-1}e^{2\pi i lm/Q}|\Psi_l(\lambda_1,\ldots,\lambda_N)\rangle \qquad (2.55)$$

$$\langle\Phi_m(\lambda_1,\ldots,\lambda_N)| = \frac{1}{\sqrt{Q}}\sum_{l=0}^{Q-1}\langle\Psi_l(\lambda_1,\ldots,\lambda_N)|e^{-2\pi i lm/Q}. \qquad (2.56)$$

By means of relations (2.45)–(2.48) they are shown to be the eigenvectors of the transfer matrix:

$$t(\mu)|\Phi_m(\lambda_1,\ldots,\lambda_N)\rangle = \Lambda_m(\mu;\{\lambda_k\})|\Phi_m(\lambda_1,\ldots,\lambda_N)\rangle \qquad (2.57)$$

$$\langle\Phi_m(\lambda_1,\ldots,\lambda_N)|t(\mu) = \langle\Phi_m(\lambda_1,\ldots,\lambda_N)|\Lambda_m(\mu;\{\lambda_k\}) \qquad (2.58)$$

if the spectral parameters $\{\lambda_j\}$ satisfy the *Bethe ansatz equations*:

$$\left(\frac{h(\lambda_j+\eta)}{h(\lambda_j-\eta)}\right)^L = e^{-4\pi i m/Q}\prod_{k\neq j}^{N}\frac{\alpha(\lambda_k,\lambda_j)}{\alpha(\lambda_j,\lambda_k)} \qquad (j=1,\ldots,N). \qquad (2.59)$$

Here the eigenvalue is given by

$$\Lambda_m(\mu;\{\lambda_k\}) = e^{2\pi i m/Q}{}_1\Lambda(\mu;\{\lambda_k\}) + e^{-2\pi i m/Q}{}_2\Lambda(\mu;\{\lambda_k\}). \qquad (2.60)$$

We thus have obtained the eigenvectors for the XYZ spin chain (2.55) and (2.56).

In the case $Q = 2$ the Bethe ansatz equations break up into N independent equations for the spectral parameters $\{\lambda_j\}$. This case corresponds to the Ising, dimer and free-fermion models [1].

3. Gaudin hypothesis

In this section we compute the sum of norms of the Bethe vectors:

$$\mathcal{M}_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Q} \sum_{l=0}^{Q-1} \langle \Psi_l^n(\lambda_1, \ldots, \lambda_n) | \Psi_l^n(\lambda_1, \ldots, \lambda_n) \rangle. \tag{3.1}$$

Here the Bethe vectors are redefined by

$$|\Psi_l^n(\lambda_1, \ldots, \lambda_n)\rangle = B_{l+N-n+1, l-N+n-1}(\lambda_1) \cdots B_{l+N, l-N}(\lambda_n) | l - N \rangle \tag{3.2}$$

$$\langle \Psi_l^n(\lambda_1, \ldots, \lambda_n)| = \langle l - N + 1 | C_{l+N-1, l-N+1}(\lambda_n) \cdots C_{l+N-n, l-N+n}(\lambda_1) \tag{3.3}$$

and the spectral parameters $\{\lambda_j\}$ are supposed to satisfy the Bethe ansatz equations:

$$r(\lambda_j) \prod_{k \neq j}^{n} \frac{\alpha(\lambda_j, \lambda_k)}{\alpha(\lambda_k, \lambda_j)} = \mathrm{e}^{-4\pi \mathrm{i} m/Q} \qquad (j = 1, \ldots, n) \tag{3.4}$$

where

$$r(\lambda) = \left(\frac{h(\lambda + \eta)}{h(\lambda - \eta)} \right)^L. \tag{3.5}$$

We compute $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$ by induction on n. Let

$$\|\lambda_1, \ldots, \lambda_n\|_n = \frac{(-h'(0))^n \mathcal{M}_n(\lambda_1, \ldots, \lambda_n)}{c_L (h(2\eta))^n \prod_{j=1}^n (h(\lambda_j + \eta) h(\lambda_j - \eta))^L \prod_{j \neq k}^n \alpha(\lambda_j, \lambda_k)} \tag{3.6}$$

with the norm of the generating vectors:

$$c_L = \langle l | l - 1 \rangle = \left(\frac{2g(\eta - \frac{1}{2}(s - t))}{g(K)} \right)^L \prod_{i=1}^{L} g(\tau_{i+l-2}). \tag{3.7}$$

Notice that c_L is independent of l due to the periodicity of $g(\mu)$.

Extending Korepin's proof of the Gaudin hypothesis [5] we prove that $\|\lambda_1, \ldots, \lambda_n\|_n$ is expressed in the form of a Jacobian (see (3.15)). This result implies the Gaudin hypothesis for the XYZ spin chain; the Gaudin hypothesis is regarded as a theorem that holds for the Bethe vectors by virtue of the fact that they correspond to the eigenvectors in the usual algebraic Bethe ansatz.

Using the solutions of the Bethe ansatz equations $\{\lambda_k\}$ we introduce new parameters:

$$X_j = \frac{\mathrm{d}}{\mathrm{d}\lambda_j} \log r(\lambda_j) \qquad (j = 1, \ldots, n). \tag{3.8}$$

Lemma 1. $\|\lambda_1, \ldots, \lambda_n\|_n$ *is invariant under simultaneous replacements:*

$$\lambda_j \leftrightarrow \lambda_k \qquad \textit{and} \qquad X_j \leftrightarrow X_k \qquad (j, k = 1, \ldots, n).$$

Proof. Because of commutation relations (2.32) and (2.33), $\mathcal{M}_n(\lambda_1, \ldots, \lambda_n)$ and therefore $\|\lambda_1, \ldots, \lambda_n\|_n$ are invariant under the replacements. □

Lemma 2. $\|\lambda_1, \ldots, \lambda_n\|_n = 0$ *if* $X_1 = \cdots = X_n = 0$.

Proof. Let $4\varepsilon = \min_{j\neq k} |\lambda_j - \lambda_k|$ and consider a new continuous function $\tilde{r}(\lambda)$ that coincides with $r(\lambda_j)$ for $|\lambda - \lambda_j| \leqslant \varepsilon$ $(j, k = 1, \ldots, n)$. By definition the set $\{X_j\}$ derived from $\tilde{r}(\lambda)$ satisfies $X_1 = \cdots = X_n = 0$. Next, we introduce new spectral parameters

$$\tilde{\lambda}_j = \lambda_j + \delta \qquad |\delta| < \varepsilon \qquad (j = 1, \ldots, n). \tag{3.9}$$

These spectral parameters $\{\tilde{\lambda}_j\}$ still obey the Bethe ansatz equations (3.4), because $\alpha(\tilde{\lambda}_j, \tilde{\lambda}_k)$ depends only on $\lambda_j - \lambda_k$ and $\tilde{r}(\tilde{\lambda}_j)$ is equal to $r(\lambda_j)$ by definition of $\tilde{r}(\lambda)$ and $\{\tilde{\lambda}_j\}$ $(j, k = 1, \ldots, n)$. We define

$$F_n(\delta) = \frac{1}{Q}\sum_{l=0}^{Q-1} \langle \Psi_l^n(\lambda_1, \ldots, \lambda_n)|\Psi_l^n(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)\rangle. \tag{3.10}$$

Evaluating $F_n(\delta)$ helps us to prove the lemma. Compute

$$\frac{1}{Q}\sum_{l=0}^{Q-1}\Big(\mathrm{e}^{2\pi\mathrm{i}m/Q}\langle \Psi_{l-1}^n(\lambda_1, \ldots, \lambda_n)|A_{l+N-n,l-N+n}(\mu)|\Psi_l^n(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)\rangle$$
$$+\mathrm{e}^{-2\pi\mathrm{i}m/Q}\langle \Psi_{l+1}^n(\lambda_1, \ldots, \lambda_n)|D_{l+N-n,l-N+n}(\mu)|\Psi_l^n(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)\rangle\Big)$$

in two ways that both of $A_{l+N-n,l-N+n}(\mu)$ and $D_{l+N-n,l-N+n}(\mu)$ operate to the left or to the right. It thus follows that

$$(\Lambda_m(\mu; \{\lambda_k\}) - \Lambda_m(\mu; \{\tilde{\lambda}_k\}))F_n(\delta) = 0. \tag{3.11}$$

Since $\Lambda_m(\mu; \{\lambda_k\})$ is a continuous function for $\{\lambda_k\}$, $F_n(\delta)$ must be 0. Due to the definition of $\|\lambda_1, \ldots, \lambda_n\|_n$ the proof is complete. □

Lemma 3. $\|\lambda_1, \ldots, \lambda_n\|_n$ *satisfies a recursion relation:*

$$\|\lambda_1, \ldots, \lambda_n\|_n = \|\lambda_2, \ldots, \lambda_n\|_{n-1}^{\mathrm{mod}} X_1 + V_1 \tag{3.12}$$

where V_1 *is independent of* X_1. $\|\lambda_2, \ldots, \lambda_n\|_{n-1}^{\mathrm{mod}}$ *is defined by* $n-1$ *solutions of the Bethe ansatz equations and* $r(\lambda)$ *is modified by*

$$r^{\mathrm{mod}}(\lambda) = r(\lambda)\frac{\alpha(\lambda, \lambda_1)}{\alpha(\lambda_1, \lambda)}. \tag{3.13}$$

Proof. $\mathcal{M}_n$ is reduced to $\mathcal{M}_{n-1}$ with the help of the commutation relation (2.39) and relations (2.45) and (2.46). Letting both $A_{l+N-n+2,l-N+n}$ and $D_{l+N-n+1,l-N+n-1}$ act on the right Bethe vector we obtain

$$\mathcal{M}_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{Q}\sum_{l=0}^{Q-1}\lim_{\lambda_1^C\to\lambda_1}[\beta_{l+N-n+1}(\lambda_1^C, \lambda_1)_1\Lambda(\lambda_1^C; \{\lambda_k\}_{k\neq1})_2\Lambda(\lambda_1; \{\lambda_k\}_{k\neq1})$$
$$-\beta_{l-N+n-1}(\lambda_1^C, \lambda_1)_1\Lambda(\lambda_1; \{\lambda_k\}_{k\neq1})_2\Lambda(\lambda_1^C; \{\lambda_k\}_{k\neq1})]$$
$$\times\langle \Psi_l^{n-1}(\lambda_2, \ldots, \lambda_n)|\Psi_l^{n-1}(\lambda_2, \ldots, \lambda_n)\rangle + \text{ terms independent of } X_1$$
$$= h(2\eta)(h(\lambda_1+\eta)h(\lambda_1-\eta))^L\prod_{j\neq k}^{n}\alpha(\lambda_j, \lambda_k)$$
$$\times\frac{1}{-h'(0)}\frac{\partial}{\partial\lambda_1}\log\left(r(\lambda_1)\prod_{k=2}^{n}\frac{\alpha(\lambda_1, \lambda_k)}{\alpha(\lambda_k, \lambda_1)}\right)\mathcal{M}_{n-1}^{\mathrm{mod}}(\lambda_2, \ldots, \lambda_n)$$
$$+ \text{ terms independent of } X_1. \tag{3.14}$$

Here we have used l'Hospital's rule. Notice that extra terms whose right Bethe vectors still contain λ_1 do not generate X_1, because it raises only in the case where both of the Bethe vectors depend on λ_1 and l'Hospital's rule is applied. Formula (3.14) implies the lemma. □

Lemma 4. $\|\lambda_1\|_1 = X_1$.

Proof. The proof is straightforward with $\tau_{l+N} = \tau_{l-N}$. □

By lemmas 1–4, $\|\lambda_1, \ldots, \lambda_n\|_n$ is determined uniquely. The following is a main result of this paper and corresponds to the Gaudin hypothesis for the XYZ spin chain.

Theorem. $\|\lambda_1, \ldots, \lambda_n\|_n$ *has the following Jacobian form:*

$$\|\lambda_1, \ldots, \lambda_n\|_n = \det_n \frac{\partial \varphi_k}{\partial \lambda_j} \tag{3.15}$$

where

$$\varphi_k = \log\left(r(\lambda_k) \prod_{i \neq k}^{n} \frac{\alpha(\lambda_k, \lambda_i)}{\alpha(\lambda_i, \lambda_k)} \right). \tag{3.16}$$

Proof. It is obvious that this expression satisfies lemma 1–4. We prove its converse by induction on n. Let

$$\Delta_q = \|\lambda_1, \ldots, \lambda_q\|_q - \det_q \frac{\partial \varphi_k}{\partial \lambda_j} \qquad (q = 1, \ldots, n). \tag{3.17}$$

By lemma 4 it follows that $\Delta_1 = 0$. Let us assume that $\Delta_q = 0$ for $q = 1, \ldots, n-1$. By lemma 3 we have

$$\frac{\partial \Delta_n}{\partial X_1} = \|\lambda_2, \ldots, \lambda_n\|_{n-1}^{\text{mod}} - \det_{n-1} \frac{\partial \varphi_k^{\text{mod}}}{\partial \lambda_j}. \tag{3.18}$$

By the assumption of induction the right-hand side is equal to 0. Thus Δ_n is independent of X_1. By lemma 1 Δ_n does not depend on any X_j $(j = 1, \ldots, n)$. Hence we obtain $\Delta_n = 0$ owing to lemma 2. The proof has been completed. □

The function φ_k is expanded as

$$\begin{aligned}\varphi_k = 2\pi \mathrm{i} l_k - L\left[\pi \mathrm{i}\left(1 + \frac{\lambda_k}{K}\right) - 2\sum_{m=1}^{\infty} \frac{\sin(m\pi\lambda_k/K)\sin(m\pi(\eta - \frac{1}{2}\mathrm{i}K')/K)}{m\sinh(m\pi K'/2K)}\right] \\ - \sum_{i\neq k}^{n}\left[\pi \mathrm{i}\left(1 + \frac{\lambda_i - \lambda_k}{K}\right)\right. \\ \left. -2\sum_{m=1}^{\infty} \frac{\sin(m\pi(\lambda_i - \lambda_k)/K)\sin(m\pi(2\eta - \frac{1}{2}\mathrm{i}K')/K)}{m\sinh(m\pi K'/2K)}\right]\end{aligned} \tag{3.19}$$

where l_k is half-integer. Because of the condition for η (2.3) this series converge absolutely provided that

$$\mathrm{Im}\,\frac{\lambda_k}{K} = 0 \qquad (k = 1, \ldots, n). \tag{3.20}$$

4. Concluding remarks

We have computed the sum of norms of the Bethe vectors and have proved that it is expressed in the form of a Jacobian (3.15). Note that the Bethe vectors correspond to the eigenvectors in the usual algebraic Bethe ansatz. Our result is thus equivalent to the Gaudin hypothesis for the XYZ spin chain.

Physically, calculation of norms of the eigenvectors is important. However, it is impossible to compute them in the framework of the original generalized algebraic Bethe ansatz, because extra scalar products of the Bethe vectors with different l such that $\langle\Psi_l|\Psi_{l'}\rangle$ $(l \neq l')$ always appear, and they cannot be calculated. It is necessary to develop a new method to obtain not only norms of the eigenvectors but also scalar products of arbitrary vectors for the XYZ spin chain.

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